Indian Statistical Institute
Semestral Examination
Topology III - MMath II

Max Marks: 60

Time: 180 minutes.

Give proper and complete justification(s) for your answers. In the sequel all maps are continous, all manifolds are closed (compact without boundary) and connected. Answer any 5 from the questions below.

(1) Let X be a CW-complex and X^p its p-skeleton. Show that the inclusion map $j : X^p \hookrightarrow X$ induces isomorphism $j_* : H_i(X^p; \mathbb{Z}) \longrightarrow H_i(X; \mathbb{Z})$ for all $i \leq p-1$. Give an example to show that $j_* : H_p(X^p; \mathbb{Z}) \longrightarrow H_p(X; \mathbb{Z})$ need not be an isomorphism. [10+2]

(2) Compute $H_i(X; G), H^i(X; G)$ where $X = S^2 \times \mathbb{R}P^2$ and $G = \mathbb{Z}, \mathbb{Z}_2$. [12]

(3) Suppose that X, Y are manifolds. Show that $X \times Y$ is orientable if and only if both X and Y are orientable. [12]

(4) Discuss the notion of the degree of a map between two orientable n-manifolds. If X is an orientable n-manifold show that there is a map $f : X \longrightarrow S^n$ of degree 1. [4+8]

(5) Prove that the spaces $X = S^2 \vee S^4$ and $Y = \mathbb{C}P^2$ are not homotopically equivalent. [12]

(6) Show that the $\mathbb{Z}_2$-cohomology algebra $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$ of $\mathbb{R}P^n$ is isomorphic to the truncated polynomial algebra $\mathbb{Z}_2[a]/a^{n+1}$. Show that if X is a simply connected space and $f : X \longrightarrow \mathbb{R}P^n$ is a map, then the homomorphism

$$f^* : H^n(\mathbb{R}P^n; \mathbb{Z}_2) \longrightarrow H^n(X; \mathbb{Z}_2)$$

is the zero homomorphism. [8+4]